# ON THE TRANSFORMATION GROUP OF A
# REAL HYPERSURFACE

BY

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ABSTRACT. The group of biholomorphic transformations leaving fixed a
strongly pseudoconvex real hypersurface in a complex manifold is a Lie
group. In this paper it is shown that the Chern-Moser invariants must vanish
if this group is noncompact and the hypersurface is compact. Also consid-
ered are transformation groups of flat hypersurfaces and intransitive groups.

**Introduction.** The purpose of this paper is to study the group of structure
preserving, or pseudoconformal, transformations of a real hypersurface $M$ (of
dimension $2n + 1$) in a complex $n + 1$ manifold, or more generally, of a
manifold with the same C-R structure. We will always assume that the C-R
structure is integrable and that the Levi form is nondegenerate. It follows
from the work of Chern and Moser [2] that this group is a Lie transformation
group.

There is some similarity between the geometry of $M$ and the conformal
geometry of a Riemannian manifold. Over $M$ there is a principal fibre bundle
$B$ (the pseudoconformal bundle), a connection on $B$, and curvature invariants
the most important of which is the fourth order curvature tensor $S^{\alpha}_{\beta\rho\bar{\sigma}}$ (see [2]).
These have as analogues in the Riemannian case the bundle of conformal
frames the conformal connection, and the Weyl conformal curvature tensor.
If $n \geqslant 2$, $S$ vanishes if and only if $M$ is pseudoconformally flat, i.e., locally
equivalent to the standard sphere $S^{2n+1}$ in $\mathbb{C}^{n+1}$ [8].

Locally the structure of $M$ is given by a real one-form $\theta$, $n$ complex
one-forms $\theta^{\alpha}$, and their complex conjugates $\theta^{\bar{\alpha}} = \overline{\theta^{\alpha}}$. They satisfy

$$\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \theta^{\bar{1}} \wedge \cdots \wedge \theta^{\bar{n}} \neq 0$$

and the equation

$$d\theta \equiv ig_{\alpha\bar{\beta}}\theta^{\alpha} \wedge \theta^{\bar{\beta}} \qquad \mod \theta.$$

Here and throughout this paper Greek indices run from 1 to $n$ and the
summation convention is used. The nondegenerate hermitian matrix $(g_{\alpha\bar{\beta}})$ is

the Levi form. If $M$ is strongly pseudoconvex $\theta$ is chosen so that this matrix is negative definite. The holomorphic tangent bundle $H(M)$ is the annihilator of $\theta$ in the real tangent bundle of $M$. We will have need of the dual vector fields $X_\alpha$, $X_{\bar\alpha} = \bar{X}_\alpha$, and $X = \bar{X}$.

The real one-form $\theta$ is determined up to multiplication by a nonzero function on $M$. For a choice of $\theta$ the pair $(M, \theta)$, called a pseudohermitian manifold in [8], is the analogue of a Riemannian structure underlying a conformal structure.

In §1 we show that if $M$ is compact, connected, and strongly pseudoconvex, and if the identity component of its pseudoconformal transformation group is noncompact, then $M$ is flat. The proof owes much to that given by Obata [7] in the conformal case.

In §2 we study the unbounded one-parameter groups of pseudoconformal transformations of $S^{2n+1}$. The main result is the same as in the conformal case [6]. This is used to show that, if $M$ is as in the preceding paragraph and has a finite fundamental group, then $M$ is globally equivalent to $S^{2n+1}$. In §3 we consider one-parameter groups with fixed points on a flat space.

We turn to intransitive groups of pseudohermitian transformations in §4. The theorem here is that if $G$ is an intransitive group of transformations of $(M, \theta)$, $M$ having dimension $2n + 1$, then $G$ has dimension less than or equal to $n^2 + 1$. For $n = 1$, Cartan [1] has shown that a three dimensional group of pseudoconformal transformations of a real hypersurface in $\mathbf{C}^2$ must be transitive.

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## 1. Spaces admitting a noncompact connected Lie group of pseudoconformal transformations. In this section we prove the following:

THEOREM (1.1). *Let $M$ be a compact, connected, strongly pseudoconvex abstract real hypersurface, and let $G$ be a connected Lie group of pseudoconformal transformations of $M$. If $G$ is noncompact then $M$ is locally equivalent to the standard real hypersphere $S^{2n+1}$ in $\mathbf{C}^{n+1}$ in the pseudoconformal sense.*

This will be proved as follows. Suppose $M$ is not locally $S^{2n+1}$; then we will show that the closure of every one-parameter subgroup of $G$ is compact. By a theorem of Montgomery and Zippin [5] $G$ must itself be compact.

We first assume that $\dim M = 2n + 1$, $n \geqslant 2$.

Let $G_1$ be a one-parameter subgroup of $G$ with infinitesimal generator $Y$ on $M$. Assuming $M$ is not flat, it follows that the pseudoconformal curvature tensor $S_{\beta\rho\bar\alpha\bar\sigma}$ is nonzero somewhere. Let $U$ be a nonempty connected component of the open set where $S$ does not vanish. $U$ is invariant under $G$, since $G$ is connected.

Since $M$ is strongly pseudoconvex, we can choose a globally defined nonvanishing one-form $\theta$ annihilating the holomorphic tangent bundle $H(M)$. As in [8, §3], the one-form

$$(1.1) \qquad \theta^* = \|S\|_\theta \cdot \theta$$

is a pseudoconformal invariant on $U$. Let $X^*$ be the corresponding invariant transversal [8]. The following Lie derivatives then vanish

$$(1.2) \qquad L_Y \theta^* = 0, \qquad L_Y X^* = [Y, X^*] = 0.$$

On $U$ we can decompose $Y$ uniquely as

$$(1.3) \qquad Y = \eta X^* + \tilde{Y},$$

where

$$(1.4) \qquad \eta =. \theta^*(Y), \quad \theta^*(\tilde{Y}) = 0.$$

By the following proposition we may assume that $\eta$ does not vanish identically.

PROPOSITION (1.2). *Let $Y$ be an infinitesimal pseudoconformal transformation on a nondegenerate, integrable C-R manifold $M$. If $Y_p$ belongs to $H_p(M)$ for all $p$ in an open subset $U$ of $M$, then $Y$ vanishes identically.*

PROOF. Choose an local frame $(X, X_\alpha, X_{\bar{\alpha}})$ and dual coframe $(\theta, \theta^\alpha, \theta^{\bar{\alpha}})$. We have $d\theta = ig_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + \theta \wedge \phi$, and we can put $Y = \xi^\alpha X_\alpha + \xi^{\bar{\alpha}} X_{\bar{\alpha}}$ in the open set $U$. Since $Y$ preserves the structure, we have

$$(1.5) \qquad L_Y \theta = u\theta.$$

We also have

$$(1.6) \qquad L_Y \theta = \iota_Y d\theta + d\iota_Y \theta = ig_{\alpha\bar{\beta}}\xi^\alpha \theta^{\bar{\beta}} - ig_{\alpha\bar{\beta}}\xi^{\bar{\beta}}\theta^\alpha - \phi(Y)\theta.$$

Comparing (1.5) and (1.6) we see that, for all $\beta$, $g_{\alpha\bar{\beta}} \xi^\alpha = 0$. Since $g_{\alpha\bar{\beta}}$ is nondegenerate, $\xi^\alpha = 0$ for all $\alpha$, and so $Y$ vanishes on $U$. The flow $G_1$ of $Y$ induces a flow $G_1'$ on the pseudoconformal bundle $B$ over $M$ which preserves the connection forms. $G_1'$ acts trivially on the part of $B$ over $U$, so by Theorem (3.2) of [3], $G_1'$ consists of the identity alone. Therefore, $G_1$ is also the identity, and $Y$ vanishes everywhere on $M$. □

Returning to the proof of Theorem (1.1), we may assume that $\eta > 0$ somewhere by replacing $Y$ by $-Y$, if necessary. For a sufficiently small constant $\varepsilon > 0$ we define a nonempty closed subset of $U$ by

$$U_\varepsilon = \{ p \in U : \eta(p) \geqslant \varepsilon \}.$$

By (1.1) and (1.4) and the definition of $U$ it is seen that $\eta(p)$ goes to zero as $p$ approaches the boundary of $U$. Therefore, $U_\varepsilon$ is a closed subset of $M$ and so is compact.

$U_\varepsilon$ is invariant under $G_1$, the flow of $Y$, because

$$Y\eta = (L_Y\theta^*)(Y) + \theta^*([\,Y, Y\,]) = 0.$$

It is also invariant under the closure $\overline{G_1}$.

Let $P$ be the pseudohermitian bundle over $(U, \theta^*)$ with fibre $U(n)$ [8, §1]. $P_\varepsilon$, the part of $P$ lying over $U_\varepsilon$, is compact and invariant by $\overline{G}_1$. By Theorem (3.2) of [3] and the fact that $\overline{G}_1$ is a Lie group, $\overline{G}_1$ imbeds as a closed submanifold of $P_\varepsilon$. Therefore, $\overline{G}_1$ is compact, and Theorem (1.1) is true for $n > 2$.

For the case $n = 1$, we replace $\|S\|_\theta$ by the relative invariant $Q$ on $M^3$, the vanishing of which is necessary and sufficient for $M^3$ to be flat [1]. The same argument shows that Theorem (1.1) holds in this case also.

## 2. Unbounded one-parameter groups on the real hypersphere.

To study further compact, strongly pseudoconvex spaces $M$ admitting a noncompact, connected Lie group of pseudoconformal transformations, we can restrict ourselves to the pseudoconformally flat spaces. It follows from the structure equations of [2, §5] that $M$ can be developed locally onto the real hypersphere $S^{2n+1}$. If $M$ is also simply connected, then the same monodromy argument as in the conformal case [4] shows that $M$ can be developed globally onto an open subset of $S^{2n+1}$. This development may not be one-to-one.

A pseudoconformal vector field on $M$ will under this development be mapped to a pseudoconformal vector field on an open subset of $S^{2n+1}$. Because $S^{2n+1}$ has such a large group of transformations, any such local vector field extends to a unique global vector field. We therefore study one-parameter groups, particularly those with noncompact closure, on $S^{2n+1}$. This is a matter of linear algebra.

In this section we will prove the following:

THEOREM (2.1). *Let $G_1$ be a one-parameter group of pseudoconformal transformations of $S^{2n+1}$ having a noncompact closure. Let $F_t$ be its flow. Then either*

(1) *$F_t$ has precisely two fixed points, $p_-$ and $p_+$, such that for any other point $p$ in $S^{2n+1}$,*

$$\lim_{t \to +\infty} F_t(p) = p_+, \qquad \lim_{t \to -\infty} F_t(p) = p_-;$$

*or*

(2) *$F_t$ has precisely one fixed point $p'$, such that for any point $p$ in $S^{2n+1}$,*

$$\lim_{t \to \pm\infty} F_t(p) = p'.$$

Before giving the proof we will derive the following corollary.

COROLLARY (2.2). *Let $M$ be a compact, connected, strongly pseudoconvex C-R manifold with finite fundamental group. If $M$ admits a noncompact, connected Lie group $G$ of pseudoconformal transformations, then $M$ is globally equivalent to $S^{2n+1}$.*

PROOF. By Theorem (1.1), $M$ and hence its universal covering space $\tilde{M}$ are flat. Since $M$ is compact, the development map is a covering. Thus, $\tilde{M}$ is globally equivalent to $S^{2n+1}$. Since $G$ is noncompact, it has a closed, noncompact one-parameter subgroup $G_1$; see [5]. $G_1$ lifts to a closed noncompact one-parameter subgroup $\tilde{G}_1$ on $\tilde{M}$. By Theorem (2.1), $\tilde{G}_1$ has either one or two fixed points, which must cover the fixed points of $G_1$. Therefore, $\tilde{M}$ is a one or two sheeted covering. If it were a two sheeted covering, both fixed points of $\tilde{G}_1$ would lie over the same fixed point of $G_1$. However, from case (1) of Theorem (2.1) these two fixed points have different character as $t \to +\infty$, one attracting and one repelling; hence, they could not cover the same fixed point of $G_1$. Therefore, $M \simeq \tilde{M} \simeq S^{2n+1}$. $\square$

We go on to prove Theorem (2.1). As in [2] we view $S^{2n+1}$ as a real hyperquadric in complex projective space. A pseudoconformal transformation of $S^{2n+1}$ is induced by a projective transformation carrying $S^{2n+1}$ into itself. Such a transformation is represented by a linear isomorphism $A$ of $C^{n+2}$, the space of homogeneous coordinates, which leaves invariant $\tilde{S}$, those coordinates of points in $S^{2n+1}$. We denote by $[Z]$ the point in projective space with homogeneous coordinates $Z$.

$\tilde{S}$ is defined by

$$(2.1) \qquad \tilde{S} = \left\{ Z \in C^{n+2} : (Z, \bar{Z}) = 0 \right\},$$

where $(\ ,^{-})$ is a nondegenerate hermitian form of signature $(n + 1, 1)$. We can view the pseudoconformal group of $S^{2n+1}$ as $SU(n + 1, 1)$.

A one-parameter subgroup $A(t)$ of $SU(n + 1, 1)$ is given by

$$(2.2) \qquad A(t) = e^{tB}.$$

where

$$(2.3) \qquad (BZ, \bar{W}) + (Z, \overline{BW}) = 0, \qquad Z, W \in C^{n+2}.$$

Eigenvectors $Z$ of $B$ on $S$ correspond to fixed points $[Z]$ of the group $A(t)$ acting on $S^{2n+1}$.

Condition (2.3) readily implies the following:

LEMMA (2.3). *Let $Z$ be an eigenvector of $B$ with eigenvalue $\lambda$. If $Z$ is not in $\tilde{S}$, then $\lambda$ is purely imaginary.* $\square$

LEMMA (2.4). *Let $(\ ,^{-})$ be an hermitian form on $C^{N+1}$, and $B$ be any operator satisfying (2.3). If $B$ has no eigenvectors on $\tilde{S}$, then $C^{N+1}$ has a basis of eigenvectors with purely imaginary eigenvalues.*

PROOF. $B$ has some eigenvector, $BZ_0 = \lambda_0 Z_0$, and $\lambda_0 + \bar{\lambda}_0 = 0$ by Lemma (2.3). By assumption $Z_0 \notin (Z_0)^{\perp}$; hence, $C^{N+1} = (Z_0) \oplus (Z_0)^{\perp}$. $(Z_0)^{\perp}$ is invariant under $B$ by (2.3). We restrict $B$ and the hermitian form to $(Z_0)^{\perp}$,

where, by induction, we have a basis $Z_1, \ldots, Z_N$ of eigenvectors with purely imaginary eigenvalues. $Z_0, Z_1, \ldots, Z_N$ is the required basis of $C^{N+1}$. $\square$

For the operator $B$ of Lemma (2.4) it follows that the corresponding one-parameter group $A(t)$ is bounded. Therefore, we assume that $B$ has at least one eigenvector on $\tilde{S}$. There are two cases.

*Case* (1). $B$ has at least two distinct eigenvectors $Z_0$ and $Z_{n+1}$ on $S$.

We must have $(Z_0, \overline{Z}_{n+1}) \neq 0$. Otherwise, $S$ would contain the complex 2-plane of all $aZ_0 + bZ_{n+1}$, $a, b \in C$, and $S^{2n+1}$ would contain a complex line, which it does not. We may assume that $(Z_0, \overline{Z}_{n+1}) = -i/2$. From (2.3), $Z_0$ and $Z_{n+1}$ have eigenvalues $\lambda$ and $-\bar{\lambda}$, respectively.

Choose $Z_1, \ldots, Z_n$, a basis of $(Z_0, Z_{n+1})^{\perp}$. If relative to the basis $(Z_0, Z_\alpha, Z_{n+1})$ of $C^{n+2}$ we have

(2.4) $$Z = \zeta^a Z_a, \qquad 0 \leqslant a \leqslant n+1,$$

then

(2.5) $$(Z, \overline{Z}) = \zeta h\,{}^t\overline{\zeta} = h_{\alpha\bar{\beta}}\zeta^\alpha\bar{\zeta}^\beta + \frac{i}{2}\left(\zeta^{n+1}\bar{\zeta}^0 - \zeta^0\overline{\zeta^{n+1}}\right),$$

where $h_{\alpha\bar{\beta}}$ is positive definite hermitian. (2.3) is now written

(2.6) $$Bh + h\,{}^t\overline{B} = 0.$$

From (2.6) it is seen that $B$ has the matrix form

(2.7) $$\begin{bmatrix} \lambda & b^\beta & p \\ c_\alpha & B_\alpha^\beta & 2ib_\alpha \\ q & \dfrac{1}{2i}c^\beta & -\bar{\lambda} \end{bmatrix},$$

where

(2.8) $$\begin{gathered} p = \bar{p}, \quad q = \bar{q}, \quad B_\alpha^\gamma h_{\gamma\bar{\beta}} + h_{\alpha\bar{\gamma}}B_{\bar{\beta}}^{\bar{\gamma}} = 0, \\ c_\alpha = h_{\alpha\bar{\gamma}}c^{\bar{\gamma}}, \quad b_\alpha = h_{\alpha\bar{\gamma}}b^{\bar{\gamma}}, \quad \text{trace } B = B_\alpha^\alpha + \lambda - \bar{\lambda} = 0. \end{gathered}$$

Since $Z_0$ and $Z_{n+1}$ are eigenvectors, we have

$$p = q = c_\alpha = c^\beta = b_\alpha = b^\beta = 0.$$

Since $h_{\alpha\bar{\beta}}$ is positive definite hermitian and the second relation of (2.8) holds, we may choose $Z_1, \ldots, Z_n$ so that both $(Z_\alpha, \overline{Z}_\beta) = \delta_{\alpha\bar{\beta}}$ and $BZ_\alpha = \lambda_\alpha Z_\alpha$ where $\lambda_\alpha + \bar{\lambda}_\alpha = 0$. As $A(t)$ is unbounded, we must have $\lambda = \mu + i\nu$, $\mu \neq 0$.

Let $Z \neq 0$ given by (2.4) represent a point $[Z]$ in $S^{2n+1}$ other than $[Z_0]$ and $[Z_{n+1}]$. From (2.5) it is seen that $\zeta^0 \neq 0$ and $\zeta^{n+1} \neq 0$. The action of $A(t)$ is given by

$$\left(\zeta^0, \zeta^\alpha, \zeta^{n+1}\right) \rightarrow \left(e^{t(\mu+i\nu)}\zeta^0, e^{i\lambda_\alpha}\zeta^\alpha, e^{-t(\mu-i\nu)}\zeta^{n+1}\right).$$

Let us assume $\mu > 0$, so that as $t \to +\infty$ we have

$$\left| e^{t(\mu+i\nu)} \zeta^0 \right| \to +\infty, \qquad \left| e^{-t(\mu-i\nu)} \zeta^{n+1} \right| \to 0,$$

and the other components remain bounded. Therefore, as $t \to +\infty$, $[Z] \to [Z_0]$. Also, as $t \to -\infty$, $[Z] \to [Z_{n+1}]$.

Case (2). $B$ has only one eigenvector $Z_0$ in $\tilde{S}$ with eigenvalue $\lambda_0$.

LEMMA (2.5). *If $B$ satisfies (2.6) and has $\lambda$ as an eigenvalue, it also has $-\bar{\lambda}$ as an eigenvalue.*

PROOF.

$$0 = \det(B - \lambda I)\det(h) = \det(Bh - \lambda h) = \det(-h^t\bar{B} - \lambda h)$$

$$= \det(-h)\det{}^t\overline{(B + \bar{\lambda}I)}. \quad \square$$

Let $W$ be an eigenvector with eigenvalue $-\bar{\lambda}_0$. We are assuming $(W, \bar{W}) \neq 0$, so that $\lambda_0$ is purely imaginary, as are all the other eigenvalues. Let $\lambda_0, \lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of $B$ with eigenspaces $K_0, K_1, \ldots, K_r$, respectively.

LEMMA (2.6). $K_0 + K_1 + \cdots + K_r \subseteq (Z_0)^\perp$.

PROOF. It follows from (2.3) that if $BZ_j = \lambda_j Z_j$ for $j > 0$, then $(Z_j, \bar{Z}_0) = 0$. Suppose $Z' \in K_0$ and $(Z', \bar{Z}) \neq 0$, say $(Z', \bar{Z}_0) = 1$. Put $Z = aZ_0 + Z'$, $a = \bar{a}$. Then $(Z, \bar{Z}) = 2a + (Z', \bar{Z}')$, and we can choose $a$ so that $Z$ lies in $\tilde{S}$, contradicting the fact that $Z_0$ is the only eigenvector in $S$. Hence $K_0 \subseteq (Z_0)^\perp$. $\square$

LEMMA (2.7). *For any $Z$ in $C^{n+2}$ the complex line determined by the origin and $A(t)Z$ approaches the subspace $K_0 + \cdots + K_r$ as $t \to \pm\infty$.*

This will finish the proof of Theorem (2.1) as follows: since $S$ is invariant by $A(t)$ and $S \cap (K_0 + \cdots + K_r) = (Z_0)$, we have, for any $[Z]$ in $S^{2n+1}$, $[A(t)Z] \to [Z_0]$ as $t \to \pm\infty$.

PROOF. This will follow from the Jordan canonical form for $B$, according to which $C^{n+2} = V^0 \oplus \cdots \oplus V^r$. The dimension $n_j$ of $V^j$ is the multiplicity of $\lambda_j$ in the characteristic polynomial of $B$, and each $V^j$ is invariant under $B$ and under $A(t)$. $B$ induces on each $V^j$ an operator $B_j$ of the form

$$B_j = N_j + D_j, \quad N_j D_j = D_j N_j, \quad D_j = \lambda_j I_{n_j},$$

where $N_j$ is nilpotent. $A(t)$ decomposes as $A(t) = e^{tB_0} \oplus \cdots \oplus e^{tB_r}$, and

$$e^{tB_j} = e^{t\lambda_j}\left( I + tN_j + \cdots + \frac{t^k}{k!}(N_j)^k \right), \qquad |e^{t\lambda_j}| = 1,$$

where $k < n_j$ is such that $(N_j)^k \neq 0$ and all higher powers of $N$ vanish.

Since any $Z$ in $C^{n+2}$ has the decomposition $Z = Z^0 + \cdots + Z^r$, $Z^j \in V^j$, it suffices to show that the complex line determined by the origin and $e^{tB_j}Z^j$ approaches $K_j$ as $t \to \pm\infty$. To simplify the notation assume $n_j = k + 1$, so that $N_j$ has all ones below the main diagonal, and $V^j$ contains a unique (up to multiple) eigenvector $W^j = (1, 0, \ldots, 0)$. Putting $Z^j = (\zeta^0, \ldots, \zeta^k)$, we have

$$\zeta \cdot e^{tN} = \left( \zeta^0 + \zeta^1 t + \cdots + \zeta^k \frac{t^k}{k!}, \zeta^1 + \cdots + \zeta^k \frac{t^{k-1}}{(k-1)!}, \right.$$
$$\left. \ldots, \zeta^{k-1} + \zeta^k t, \zeta^k \right).$$

As $t \to \pm\infty$ the dominant term is in the first coordinate; hence, the line of $\zeta \cdot e^{tN}$ approaches the line of $W^j$.  $\square$

Theorem (2.1) is proved.

**3. One parameter groups on pseudoconformally flat spaces.** In this section we will prove the following theorem.

THEOREM (3.1). *Let $M$ be connected, strongly pseudoconvex, and pseudoconformally flat, and suppose $M$ admits a one-parameter Lie group $G_1$ of transformations which has a fixed point $p_0$. Then either $G_1$ is compact or $M$ is globally equivalent to $S^{2n+1}$ or to $S^{2n+1}$ with a point deleted.*

Let $Y$ be the infinitesimal generator of $G_1$ on $M$. Develop a small neighborhood $U$ of $p_0$ onto an open subset $U'$ of $S^{2n+1}$, $p_0$ corresponding to $p_0'$. $Y|_U$ corresponds to a pseudoconformal vector field on $U'$, which extends to a unique global $Y'$ on $S^{2n+1}$. There are three cases according to whether the one-parameter group $G_1'$ of $Y'$ is

(1) bounded, i.e., has compact closure $\overline{G_1'}$ in $SU(n + 1, 1)$,

(2) unbounded with behavior (1) of Theorem (2.1), or

(3) unbounded with behavior (2) of Theorem (2.1).

*Case* (1). By integration over $\overline{G_1'}$, $\overline{G_1'}$ leaves invariant some $\theta'$ and also the corresponding positive definite Riemannian metric

$$ds^2 = \theta'^2 - \mathrm{Re}\left( g_{\alpha\bar\beta}' \theta'^\alpha \otimes \theta'^{\bar\beta} \right)$$

on $S^{2n+1}$. A small geodesic ball $B'$ in $U'$ centered at $p_0'$ is mapped isometrically by each $g'$ in $G_1'$. Each $g$ in $G_1$ leaves the corresponding $\theta$ on $U$ invariant and maps the corresponding $B \subseteq U$ diffeomorphically.

LEMMA (3.2). *In Case (1), $G_1$ is compact.*

PROOF. We introduce the following notation:

$\mathrm{PsC}(M, p_0)$ = the group of global pseudoconformal transformations of $M$ leaving $p_0$ fixed.

$\mathrm{PsC}(B, p_0)$ = the group of pseudoconformal transformations of $B$ leaving $p_0$ fixed.

$\mathrm{PsC}'(B, p_0) = \mathrm{PsC}(B, p_0) \cap \mathrm{PsC}(M, p_0)$.

$\mathrm{PsH}(B, \theta, p_0)$ = the group of pseudohermitian transformations leaving $p_0$ fixed.

$\mathrm{PsH}(B, \theta, p_0)$ is compact. We have

$$G_1 \text{ closed } \subseteq \mathrm{PsC}(M, p_0),$$

since $G_1$ is a Lie subgroup; therefore,

$$G_1 \text{ closed } \subseteq \mathrm{PsC}'(B, p_0).$$

Since

$$G_1 \subseteq \mathrm{PsH}(B, \theta, p_0) \subseteq \mathrm{PsC}(B, p_0),$$

it will suffice to show that $\mathrm{PsC}'(B, p_0)$ is a closed subset of $\mathrm{PsC}(B, p_0)$.

Let $\{\phi_j\}$ be a sequence in $\mathrm{PsC}'(B, p_0)$ and suppose $\phi_j$ converges to $\phi$ in $\mathrm{PsC}(B, p_0)$. We must show that $\phi$ is globally defined on $M$. Choose a pseudoconformal frame $e_0$ at $p_0$. We have $\phi_j \cdot e_0 \to \phi \cdot e_0$. By Theorem (3.2) of [3] the map $g \to g \cdot e_0$ embeds $\mathrm{PsC}(M)$ into the bundle $B$ of pseudoconformal frames as a closed submanifold. So there exists some $\phi_0$ in $\mathrm{PsC}(M)$ with $\phi_0 \cdot e_0 = \phi \cdot e_0$. Since $\phi$ and $\phi_0$ agree at $e_0$, we have $\phi = \phi_0$ on $B$.   $\square$

*Case* (2). Pass to the universal covering $\tilde{M}$ of $M$, $\tilde{Y}$ covering $Y$ and $p_1$ lying over $p_0$. Let $f$ be development of $\tilde{M}$ over $S^{2n+1}$. Let $p_1$ correspond to $p_-$ of Theorem (2.1) and $\tilde{Y}$ correspond to $Y'$ under $f$. Define

$$V = \{p_1\} \cup \{\text{all orbits of } \tilde{Y} \text{ leaving } p_1\}.$$

Since $Y$ and hence $\tilde{Y}$ are complete vector fields, $f$ maps $V$ one-to-one onto $S^{2n+1} - (p_+)$. Suppose $p_2$ is in $\mathrm{bdry}(V)$; then $f(p_2) = p_+$. Since $f$ is a local diffeomorphism, $\mathrm{bdry}(V)$ is discrete, and so $V$ has no exterior. Furthermore, $\tilde{Y}$ has exactly one zero of type $p_-$. Interchanging $p_2$ and $p_1$ in this argument shows that $\tilde{Y}$ has exactly one zero of type $p_+$ also. So, either $\mathrm{bdry}(V) = \{p_2\}$ and $\tilde{M} \simeq S^{2n+1}$, or $\mathrm{bdry}(V) = \varnothing$ and $\tilde{M} \simeq S^{2n+1} - \{p_+\}$. In the latter case $\tilde{Y}$ has only one zero, so $\tilde{M}$ is a one sheeted covering. In the former case $\tilde{Y}$ has two zeroes of opposite character, so again $\tilde{M}$ is a one sheeted covering.

*Case* (3). We proceed as in Case (2) except that there is only one zero $p'$ of $Y'$ and $p' = f(p_1)$. Let

$$U = \{\text{all orbits of } \tilde{Y} \text{ leaving } p_1\}.$$

Again, since $\tilde{Y}$ is complete, $f$ maps $U$ one-to-one onto $S^{2n+1} - \{p'\}$, $\mathrm{bdry}(U) \subseteq f^{-1}(p')$ is discrete, and $U$ has no exterior. For any $p_2$ in $U$ $\lim_{t \to -\infty} \phi_t(p_2) = p_1$, where $\phi_t$ is the flow of $\tilde{Y}$. Suppose also $f(p_1^*) = p'$. Then for any $p_2^*$ in

$U^*$, $\lim_{t \to -\infty} \phi_t(p_2^*) = p_1^*$. As $M$ is connected, $U \cap U^* \neq \varnothing$, so we have $p_1 = p_1^*$. Therefore, $f$ maps $\tilde{M} = U \cup \{p_1\}$ one-to-one and onto $S^{2n+1}$. Since $\tilde{Y}$ has only one zero, $M \simeq \tilde{M} \simeq S^{2n+1}$.

This proves Theorem (3.1).

In view of Theorem (3.1) it follows that if $M$ satisfies the hypothesis of Theorem (1.1), then either $M \cong S^{2n+1}$ or every closed noncompact one-parameter subgroup on $M$ has no fixed points. In this respect let us note the following:

PROPOSITION (3.3). *Suppose $G_1$ is a one-parameter Lie group of pseudoconformal transformations of $M$ as in Theorem (1.1), and let $Y$ be its infinitesimal generator. If $Y$ is never tangent to $H(M)$, then $G_1$ is compact.*

PROOF. Choose some $\theta$ defining $H(M)$. Let $\eta = \theta(Y)$. $\eta$ is never zero, since $Y$ is always transversal. We have $L_Y\theta = u\theta$, so that $L_Y\eta = u\eta$, and $L_Y(\eta^{-1}\theta) = 0$. $G_1$ leaves $\theta^* = \eta^{-1}\theta$ invariant, so must be compact by Theorem (1.2) of [8]. □

### 4. Intransitive groups. In this section we prove the following theorem.

THEOREM (4.1). *Let $G$ be an effective Lie group of pseudohermitian transformations of a strongly pseudoconvex, connected, C-R manifold $(M, \theta)$ of dimension $2n + 1$. If $G$ is not transitive then the dimension of $G$ is less than or equal to $n^2 + 1$.*

EXAMPLE (4.2). To see that the estimate on $\dim G$ is the best possible, consider $S^{2n+1} \subseteq C^{n+1}$ with $\theta = i\partial r$, where $r = z^1 z^{\bar{1}} + \cdots + z^n z^{\bar{n}} + w\bar{w} - 1$. The group is $G = U(n) + U(1)$ acting on $C^{n+1}$ by

$$(z^\alpha, w) \to (z^\beta U_\beta^\alpha, w\mu), \quad U^t \bar{U} = I, \mu\bar{\mu} = 1.$$

This group preserves $r$ and hence $\theta$ and is clearly intransitive.

To prove Theorem (4.1) we recall that relative to an admissible coframe as in [8], we have

$$(4.1) \qquad\qquad d\theta = ig_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}.$$

On $H(M)$ we have the negative definite hermitian form

$$(4.2) \qquad\qquad h = g_{\alpha\bar{\beta}}\theta^\alpha\theta^{\bar{\beta}}.$$

This form is invariant under the action of $G$. For a point $p$ in $M$ we use the notation

$$G(p) = \text{orbit of } p \text{ under } G$$

and

$$G_p = \text{isotropy subgroup of } p,$$

so that $G(p) = G/G_p$. We also define $W_p = T_p(G(p)) \cap H_p(M)$, and

$$J = \text{complex multiplication on } H_p(M).$$

$W_p$ is a real subspace of $H_p$, so $0 \leqslant \dim W_p \leqslant 2n$, and is invariant under the action of $G_p$. We have a faithful representation of $G_p$ on $U(n)$, the unitary group of $(H_p, h, J)$.

The proof is divided into four parts.

(1) Suppose $W_p = 0$ for some $p$ in $M$. Then $\dim G(p) \leqslant 1$ and $\dim G = \dim G(p) + \dim G_p \leqslant 1 + n^2$.

(2) *Claim.* For any $p$ in $M$, $\dim W_p < 2n$, i.e., $W_p \neq H_p$, If $W_p = H_p$, then two cases are possible:

(a) $T_p(G(p)) = T_p(M)$. Then $G$ is locally transitive at $p$ and $G(p)$ is an open subset of $M$. Since the pseudohermitian bundle has compact fibres $U(n)$ and $G$ is a Lie transformation group, $G(p)$ is also closed in $M$. So $G(p) = M$, contradicting the fact that $G$ is not transitive.

(b) $T_p(G(p)) = H_p(M)$. Then $G(p)$ is a maximal integral submanifold of $\theta = 0$. Therefore $d\theta$ vanishes when restricted to $G(p)$, contradicting the fact that the form (4.2) is negative definite on $H_p(M)$.

So we may assume that $0 < \dim W_p < 2n$ for all $p$ in $M$. Let us denote $V_p = W_p \cap JW_p$, and $K_p = W_p + JW_p$. These are $G_p$ invariant complex subspaces of $H_p(M)$.

(3) Suppose that, for some $p$ in $M$, $W_p$ is not a totally real subspace of $H_p$, i.e., $V_p \neq 0$. We get a nontrivial $G_p$ invariant decomposition $H_p = V_p \oplus V_p^{\perp}$. Letting $l = \dim_C V_p$, we have $G_p \subseteq U(l) \oplus U(n - l)$. So

$$\dim G_p \leqslant l^2 + (n - l)^2 \leqslant 1 + (n - 1)^2,$$

and

$$\dim G \leqslant 2n + 1 + (n - 1)^2 = n^2 + 2.$$

If equality holds in the last estimate, then $\dim G(p) = 2n$ and $\dim G_p = 1 + (n - 1)^2$. The complex dimension of $V_p$ is then $n - 1$, and $G_p \simeq U(n - 1) \oplus U(1)$. Since the dimension of $W_p$ is at most one less than $\dim G(p)$, we have $\dim W_p = 2n - 1$. We have $G_p$ invariant decompositions $W_p = V_p \oplus R$, and $H_p = V_p \oplus (R + JR)$, where $R$ is a one dimensional subspace. It follows that $G_p \subseteq U(n - 1) \oplus O(1, R)$, implying that $\dim G_p \leqslant (n - 1)^2$, a contradiction. Hence $\dim G < n^2 + 1$.

(4) Suppose that, for some $p$ in $M$, $W_p$ is nontrivial and totally real. We then have $\dim W_p = k \leqslant n$, and $K_p$ is of complex dimension $k$. We have the $G_p$ invariant decompositions $H_p = K_p \oplus K_p^{\perp}$, and $K_p = W_p \oplus JW_p$. Therefore,

$$\dim G_p \leqslant \tfrac{1}{2}k(k - 1) + (n - k)^2,$$

and

$$\dim G \leqslant (k + 1) + \tfrac{1}{2} k(k - 1) + (n - k)^2$$
$$\leqslant n^2 + 1 - \tfrac{1}{2} k^2 + \tfrac{1}{2} k \leqslant n^2 + 1,$$

since $k < n$.

Theorem (4.1) is proved.

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